A generalized Brown-Proschan model for preventive and corrective maintenance

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I. Introduction

The dependability of complex repairable systems depends strongly on the efficiency of maintenance actions.

Corrective Maintenance (CM): After a failure, put the system into a state in which it can perform its function again.

Preventive Maintenance (PM): When the system is operating, intends to slow down the wear process.

Basic maintenance effect assumptions:

- **As Bad As Old (ABAO):** restores the system in the same state it was just before maintenance.
- **As Good As New (AGAN):** restores the system as if it were new.

Reality is between these two extreme cases: **Imperfect maintenance**.

Brown-Proschan model [83], CM is :

- **AGAN** with probability $\boldsymbol{p}$,
- **ABAO** with probability $\boldsymbol{1-p}$.
- repair effects (AGAN or ABAO) are mutually independent and independent of already observed failure times.

CM effects can be characterized with random variables:

$$\boldsymbol{B_i} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ CM is AGAN} \\ 0 & \text{if the } i^{\text{th}} \text{ CM is ABAO} \end{cases}$$

Statistical studies when the $\{\boldsymbol{B_i}\}_{i\geq 1}$ are known:
Whitaker-Samaniego[89], Hollander-Presnell-Sethuraman [92], Kvam-Singh-Whitaker [02], Bathe-Franz [96], Agustin-Pena[99],...

Practical purposes : $\{B_i\}_{i\geq 0}$ are hidden variables.

$\boldsymbol{p}$ characterizes **maintenance efficiency**:

- $\boldsymbol{p = 0}$: ABAO
- $\boldsymbol{p = 1}$: AGAN
- $\boldsymbol{0 < p < 1}$: imperfect

Joint assessment of CM efficiency and intrinsic wear-out:
Lim [98]; Lim-Lie[00]; Lim-Lu-Park[98]; Langseth-Lindqvist [04].

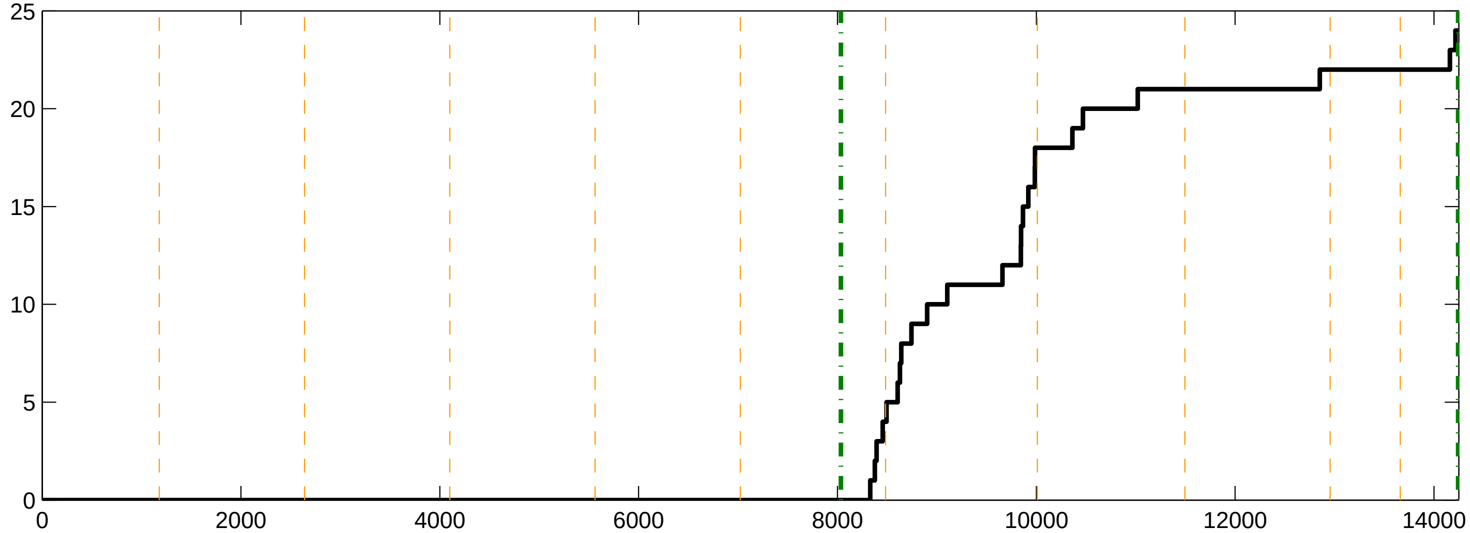
Presentation aim:

- Generalize the BP model to PM effects
- Assess PM efficiency and intrinsic wear-out when PM effects are unknown
- Compute reliability indicators.

II/ A maintenance data set

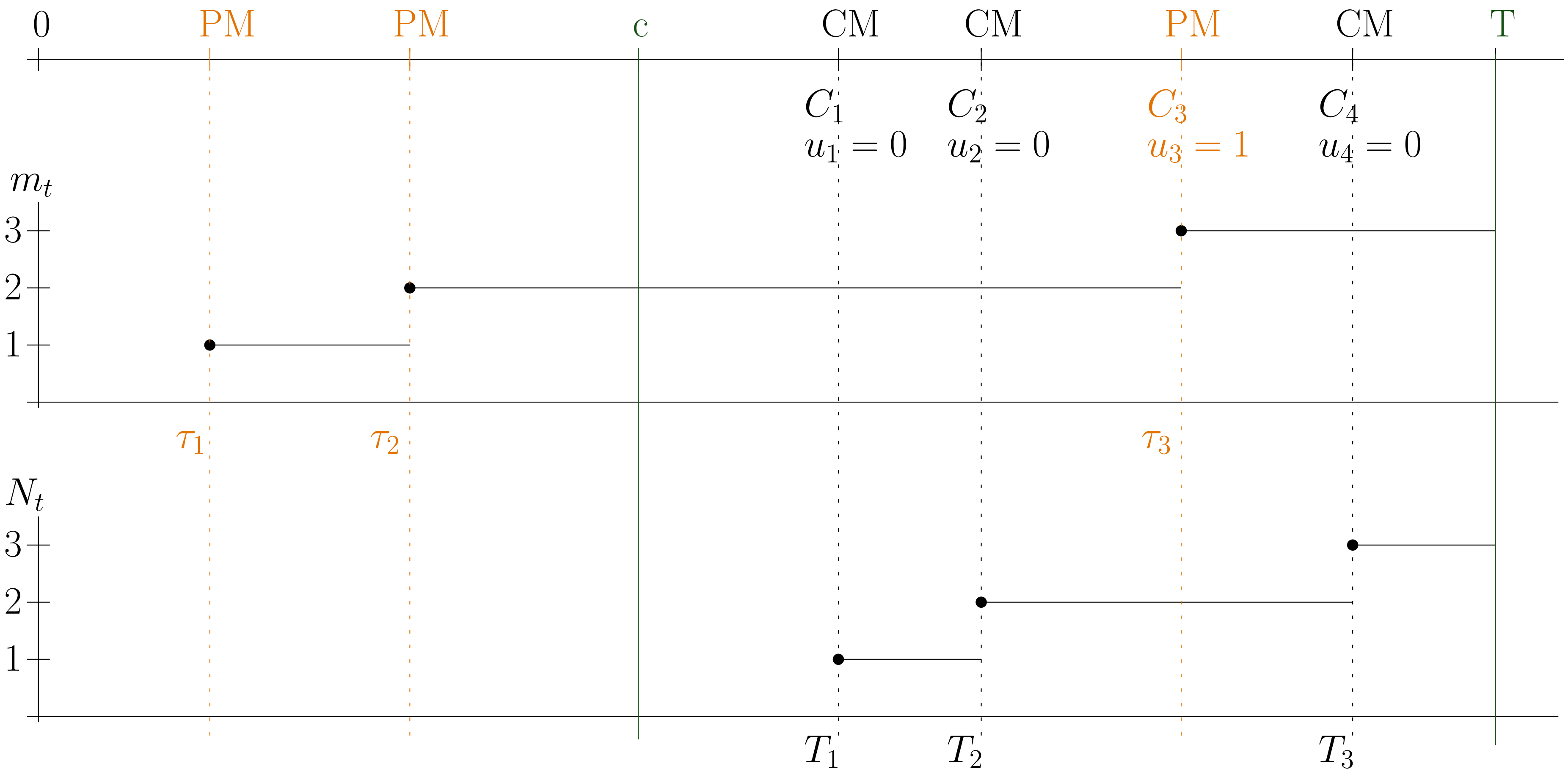
PM and CM times of a subsystem of a fossil-fired thermal plant of EDF:

1179	**2640**	**4101**	**5562**	**7023**	**8035**	8329	8376	8393	8455
8484	8494	8605	8628	8641	8744	8903	9105	9660	9845
9846	9866	9919	9985	9987	**10010**	10363	10470	11021	**11494**
12851	**12956**	**13662**	14161	14217	**14244**				



Cumulative number of failures

III. Notations



IV. Model assumptions

- **$\boldsymbol{\lambda(t)}$** the failure rate of the new unmaintained system. $\Lambda(t) = \int_0^t \lambda(s)\,ds$,
- Maintenance durations are not taken into account,
- PM are done at deterministic times, all the PM times are known
- CM are done at random times, CM are observed over $[c, T]$ $(c < T)$
- **CM** effects are **ABAO**,
- **PM** effects follow a Brown-Proschan model, i.e. PM effects are :
  - mutually independent and independent of previous failure times,
  - **AGAN** with probability $\boldsymbol{p}$,
  - **ABAO** with probability $\boldsymbol{1-p}$.

$$\boldsymbol{B_i} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ PM is AGAN} \\ 0 & \text{if the } i^{\text{th}} \text{ PM is ABAO} \end{cases}$$

$$P(B_i = 1 \mid \underline{T}_i, \underline{B}_{i-1}) = P(B_i = 1) = p$$

V. Joint assessment of intrinsic wear-out and PM efficiency
a/ Maximum likelihood method

Likelihood associated to a single observation of the failure process over $[c, t]$:

$$L_t(\theta) = f_{\underline{T}_{n_t} \,|\, N_t = n_t}(\underline{t}_{n_t}) \, P(N_t = n_t)$$

Maximum Likelihood Estimator (MLE):

$$\hat{\theta} = \arg\max_{\theta} \; L_T(\theta) = \arg\max_{\theta} \; log(L_T(\theta))$$

Notations: $L_c(\theta) = 1$

$L_{\tau,t}(\theta) = f_{\underline{T}_{n_t - n_\tau}, N_{t-\tau} = n_t - n_\tau}(t_{n_\tau + 1} - \tau, ..., t_{n_t} - \tau) \Rightarrow$ the likelihood associated to the system new in τ and observed over$[c \vee \tau, t]$.

Let us denote : $D_t^m = (1-p)^{m_{t^-}-m} \left[\prod_{c \vee \tau_m \le t_j \le t} \lambda(t_j - \tau_m) \right]$

Two equivalent equations for likelihood recursive computation:

- $L_t(\theta) = \left[\sum_{m=0}^{m_{t^-}} p^{\mathbb{1}_{\{m \neq 0\}}} D_t^m \, e^{-(\Lambda(t-\tau_m)-\Lambda((c\vee\tau_m)-\tau_m))} L_{c\vee\tau_m}(\theta) \right]$

  $\Rightarrow$ forward computation algorithm.

- $L_t(\theta) = \left[\sum_{\tau \in \{\tau_1,\ldots,\tau_{m_{t^-}},t\}} p^{\mathbb{1}_{\{\tau \neq t\}}} D_{\tau_m}^0 \, e^{-(\Lambda(\tau_m \vee c)-\Lambda(c))} L_{\tau_m,t}(\theta) \right]$

  $\Rightarrow$ backward computation algorithm.

Proof of the equation used for the forward computation:

$\underbrace{\{B_m = 1, \{B_j = 0\}_{m<j\leq m_{t-}}}_{\tau_m \text{ is the last AGAN PM since } t}\}_{0\leq m\leq m_{t-}}$ is a partition of the probability space.

Given $B_m = 1$, the CM process can be divided into 2 independent processes.

$$L_t(\theta) = \Big[\sum_{m=0}^{m_{t-}} \underbrace{p^{\mathbb{1}_{\{m\neq 0\}}}(1-p)^{m_{t-}-m}}_{P(B_m=1,\{B_j=0\}_{m<j\leq m_{t-}})}$$

$$\underbrace{\left[\prod_{c\vee\tau_m\leq t_j\leq t} \lambda(t_j-\tau_m)\right] e^{-(\Lambda(t-\tau_m)-\Lambda((c\vee\tau_m)-\tau_m))}}_{\text{Given } B_m=1,\{B_j=0\}_{m<j\leq m_{t-}},\text{ CM times after } \tau_m \text{ follow a NHPP initialized in } \tau_m}$$

$$\underbrace{L_{c\vee\tau_m}(\theta)\Big]}_{\text{Given } B_m=1,\text{ maintenances before } \tau_m \text{ follow a BP PM-ABAO CM model}}$$

b/ MLE combined with moment estimation : $\lambda(t) = \alpha\beta t^{\beta-1}$

Power Law Process: $E[N_T] = \alpha(T^\beta - c^\beta) \Rightarrow \tilde{\alpha}_\beta = N_T/(T^\beta - c^\beta)$.

BP model , $E[N_t] = \alpha\, S_t$, where

$$S_t = \sum_{\tau \in \{\tau_{m_c+1},\dots,\tau_{m_{t^-}},t\}} \sum_{k=0}^{m_{\tau^-}} p^{\mathbb{1}_{\{k\neq 0\}}}(1-p)^{m_{\tau^-}-k}((\tau-\tau_k)^\beta - ((\tau_{m_{\tau^-}} \vee c) - \tau_k)^\beta)$$

MLE combined with moment estimation of α: $\tilde{\alpha}_{\beta,p} = \dfrac{N_T}{S_T}$

$$E[\tilde{\alpha}_{\beta,p}] = \alpha$$

$$(\tilde{\beta}, \tilde{p}) = \arg\max_{(\beta,p)} log(L_T(\tilde{\alpha}_{\beta,p}, \beta, p)) \quad \text{and} \quad \tilde{\alpha} = \tilde{\alpha}_{\tilde{\beta},\tilde{p}}$$

$\Rightarrow$ Dimension reduction of the likelihood maximization space.

c/ Individual PM efficiency assessment

$\boldsymbol{p}$ characterized the **average global PM efficiency**.

The $\boldsymbol{m}^{\text{th}}$ **PM effect** can be characterized by:

$$\boldsymbol{\pi_m^\theta = P(B_m = 1 \mid N_T = n_T, \underline{T}_{n_T} = \underline{t}_{n_T})}$$

which verifies:

$$\pi_m^\theta = p \frac{L_{c \vee \tau_m}(\theta)\ L_{\tau_m,T}(\theta)}{L_T(\theta)}$$

It can naturally be estimated by: $\boldsymbol{\pi_m^{\hat{\theta}}}$

- $L_{\tau_m}(\hat{\theta})$: intermediate computing values of the forward algorithm
- $L_{\tau_m,T}(\hat{\theta})$: intermediate computing values of the backward algorithm

d/ Expectation-Maximization algorithm

Complete likelihood:

$$L_t^c(\theta) = f_{\underline{T}_{n_t} \,|\, N_t=n_t, \underline{B}_{n_t}=\underline{b}_{n_t}}(\underline{t}_{n_t})\, P(N_t = n_t | \underline{B}_{n_t} = \underline{b}_{n_t}) P(\underline{B}_{n_t} = \underline{b}_{n_t})$$

EM algorithm:

- **Expectation (E) step:** Compute $Q(\theta|\theta_k) = E^{\theta_k}\left[log(L_t^c(\theta)) \mid N\right]$
- **Maximization (M) step:** $\theta_{k+1} = \arg\max_{\theta} Q(\theta|\theta_k)$

$$\theta_k \underset{k\to\infty}{\longrightarrow} \text{local likelihood maxima}$$

Complete likelihood:

$$L_t^c(\theta) = \left[\prod_{n=1}^{N_t} \lambda(A_{t_n})\right] e^{-\int_c^t \lambda(A_s)\,ds} \quad \left[\prod_{m=1}^{m_t} p^{B_m}(1-p)^{1-B_m}\right]$$

where A_s is the virtual age, ie the time elapsed since the last perfect PM

$\forall s \in]\tau_m, \tau_{m+1}] \cap [c, T] \quad \lambda(A_s) = \prod_{i=0}^{m} [\lambda(s - \tau_{m-i})]^{\mathbb{1}_{\{B_{\tau_m}^{-i}\}}}$

where $B_{\tau_m}^{-i}$ = "at time τ_m, the last AGAN PM time is τ_{m-i}"

Complete likelihood:

$$\overbrace{L_t^c(\theta)}^{E^\theta[log(L_t^c(\theta))|\,N]=Q(p|\theta)} = \underbrace{\left[\prod_{n=1}^{N_t} \lambda(A_{t_n})\right] e^{-\int_c^t \lambda(A_s)\,ds}}_{E^\theta[log(\,.\,)|\,N]=Q_2(\lambda|\theta)} \underbrace{\left[\prod_{m=1}^{m_t} p^{B_m}(1-p)^{1-B_m}\right]}_{E^\theta[log(\,.\,)|\,N]=Q_1(p|\theta)}$$

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Complete likelihood:

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$$\forall s \in]\tau_m, \tau_{m+1}] \cap [c, T] \quad \lambda(A_s) = \prod_{i=0}^{m} [\lambda(s - \tau_{m-i})]^{\mathbb{1}_{\{B_{\tau_m}^{-i}\}}}$$

where $B_{\tau_m}^{-i}$ =“at time τ_m, the last AGAN PM time is τ_{m-i}”

- Q_2 function of $\boldsymbol{\pi_{m,i}^\theta = E^\theta\left[\mathbb{1}_{\{B_{\tau_m}^{-i}\}} \mid N_T = n_T, \underline{T}_{n_T} = \underline{t}_{n_T}\right]}$
- Q_1 function of $\pi_{m,0}^\theta = \pi_m^\theta = E^\theta\left[B_m \mid N_T = n_T, \underline{T}_{n_T} = \underline{t}_{n_T}\right]$

E step: Compute for $0 \le m \le m_T$ and $0 \le i \le m$,

$$\pi^{\theta}_{m,i} = E^{\theta}\left[\mathbb{1}_{\{B^{-i}_{\tau_m}\}} \mid N_T = n_T, \underline{T}_{n_T} = \underline{t}_{n_T}\right]$$

- $\pi_{0,0}(\theta) = 1$
- for $m \in \{1, ..., m_T\}$, $\pi^{\theta}_{m,0} = \pi^{\theta}_{m}$
- for $0 \le m < m_T$ and $0 \le i \le m$,

$$\pi^{\theta}_{m+1,i+1} = \pi^{\theta}_{m,i} - p^{1+\mathbb{1}_{\{m>i\}}}(1-p)^i \left[\prod_{(c\vee\tau_{m-i})<t_j\le\tau_{m+1}} \lambda(t_j - \tau_{m-i})\right]$$
$$e^{-(\Lambda((c\vee\tau_{m+1})-\tau_{m-i})-\Lambda((c\vee\tau_{m-i})-\tau_{m-i}))}\frac{L_{c\vee\tau_{m-i}}(\theta)L_{\tau_{m+1},T}(\theta)}{L_T(\theta)}$$

M step: $\lambda(t) = \alpha\beta t^{\beta-1}$

- $p_{k+1} = \left[\sum_{m=1}^{m_T} \pi_{m,0}^{\theta_k}\right] / m_T$

- $\beta_{k+1} = \arg\max_{\beta} \left[n_T \left(log(n_T\,\beta) - log(S_k(\beta)) \right) + (\beta - 1) \right.$
$$\left. \left[\sum_{\tau \in \{\tau_{m_c+1},\dots,\tau_{m_{T-}},T\}} \sum_{i=0}^{m_{\tau-}} \pi_{m_{\tau-},m_{\tau-}-i}^{\theta_k} \sum_{(c\vee\tau_{m_{\tau-}})<t_j\leq\tau} log(t_j - \tau_i) \right] \right]$$

with $S_k(\beta) = \displaystyle\sum_{\tau \in \{\tau_{m_c+1},\dots,\tau_{m_{T-}},T\}} \sum_{i=0}^{m_{\tau-}} \pi_{m_{\tau-},m_{\tau-}-i}^{\theta_k} \left((\tau - \tau_i)^{\beta} - ((c \vee \tau_{m_{\tau-}}) - \tau_i)^{\beta}\right)$

- $\alpha_{k+1} = n_T / S_k(\beta_{k+1})$

VI. Reliability indicators

- **Failure intensity:** $\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_{t^-} \,|\, \underline{T}_{N_{t^-}}, N_{t^-})$

$$\lambda_t = \frac{\sum_{m=0}^{m_{t^-}} D^m_{C_{K_{t^-}}} \, e^{-(\Lambda(t-\tau_m) - \Lambda((c\vee\tau_m)-\tau_m)} L_{c\vee\tau_m}(\theta)}{\sum_{m=0}^{m_{t^-}} D^m_{C_{K_{t^-}}} \, \lambda(t-\tau_m) \, e^{-(\Lambda(t-\tau_m) - \Lambda((c\vee\tau_m)-\tau_m)} L_{c\vee\tau_m}(\theta)}$$

- **Cumulative failure intensity:** $\Lambda_t = \int_0^t \lambda_s \, ds$

$$\Lambda_t = -\left[\sum_{k=1}^{K_t} log \left(\frac{L_{C_k^-}(\theta)}{L_{C_{k-1}^+}(\theta)} \right) \right] - log \left(\frac{L_{t^-}(\theta)}{L_{C_{K_t}^+}(\theta)} \right)$$

- **Reliability:** $P(T_{N_T+1} > s \,|\, \underline{T}_{N_T}, N_T) = exp(-(\Lambda_s - \Lambda_T))$

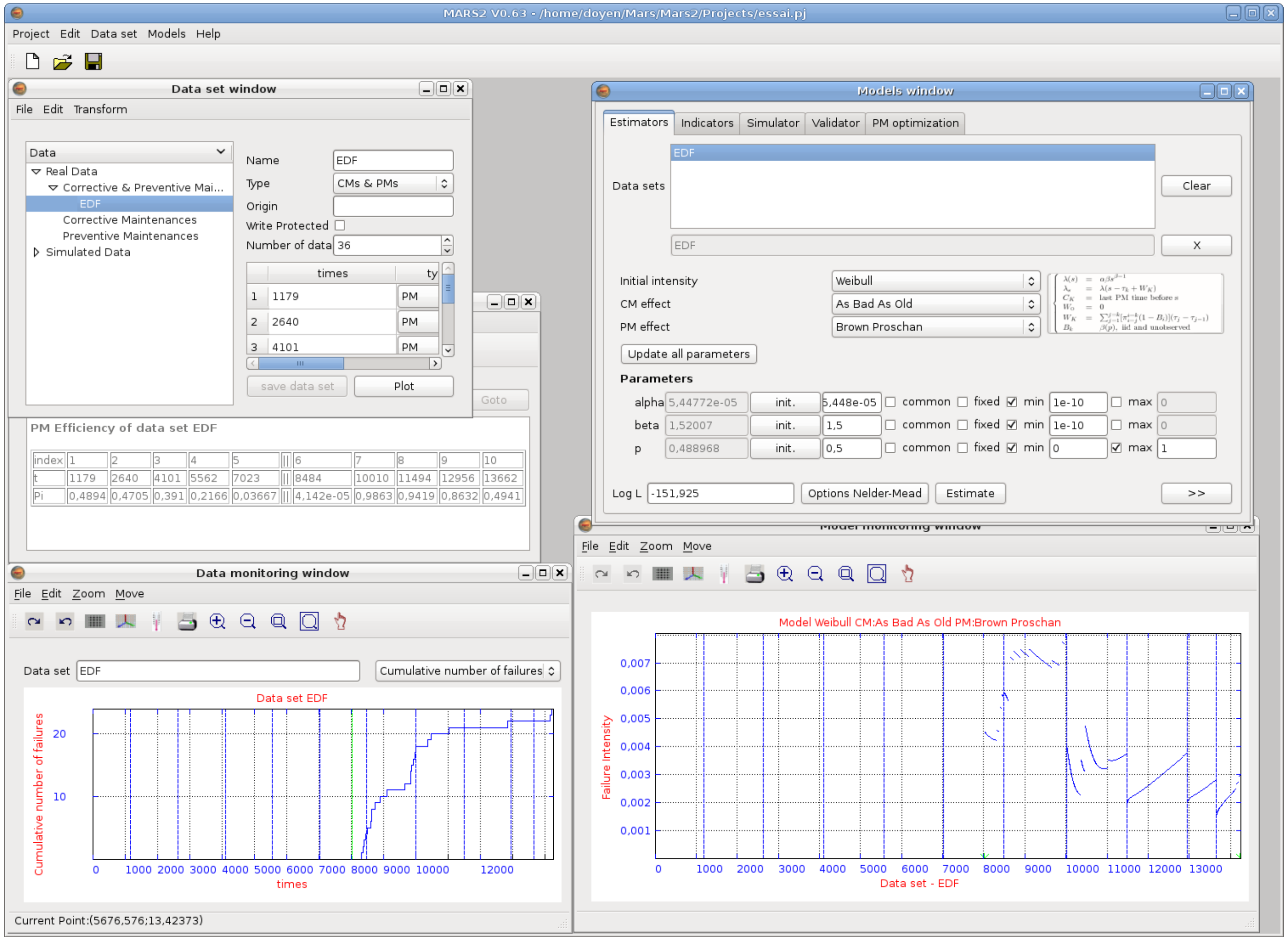
- **Expected cumulative number of failures:** $E[N_s \,|\, \underline{T}_{N_T}, N_T] =$

$$N_T + \left[\sum_{i=m_T+1}^{m_s} (\Lambda(s-\tau_i) + E[N_{\tau_i^-} \,|\, \underline{T}_{N_T}, N_T] - N_T) p(1-p)^{m_s-i} \right]$$

$$+ \left[\sum_{i=0}^{m_T} (\Lambda(s-\tau_i) - \Lambda(T-\tau_i))(1-p)^{m_s-m_T} \pi^{\theta}_{m_T, m_T-i} \right]$$

VII. Application

The BP PM-ABAO CM model is implemented in **MARS** (Maintenance Assessment of Repairable Systems): a free software developped by LJK (Grenoble university) and EDF (French electricity utility).

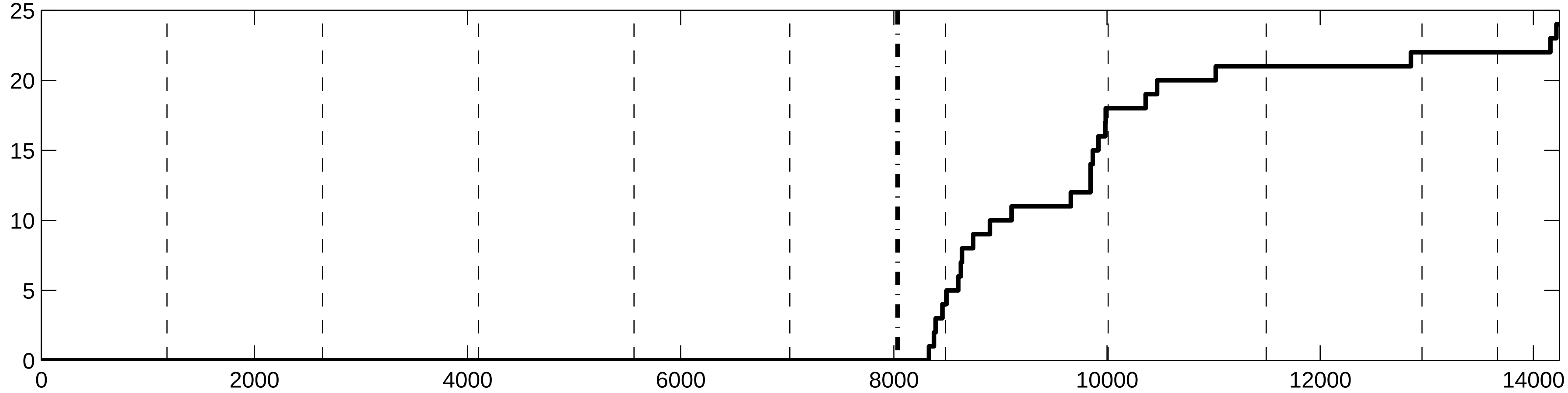


- MLE combined with moment estimation:

$$\tilde{\alpha} = 1.84 \times 10^{-6}, \quad \tilde{\beta} = 1.95, \quad \tilde{p} = 0.602, \quad log(\mathcal{L}_T(\tilde{\alpha}, \tilde{\beta}, \tilde{p})) = -150.900$$

- MLE computed with EM algorithm or direct maximization:

$$\hat{\alpha} = 1.87 \times 10^{-6}, \quad \hat{\beta} = 1.94, \quad \hat{p} = 0.614, \quad log(\mathcal{L}_T(\hat{\alpha}, \hat{\beta}, \hat{p})) = -150.902$$



$\pi_m^{\hat{\theta}}$: 0.61 0.62 0.69 0.73 0.07 0.00 1.00 0.99 0.96 0.46

ABAO ABAO AGAN AGAN AGAN

Wearing out state at time $c \Rightarrow$ failures surge

Quality of estimation: $\theta = (1.9e-6, 1.9, 0.61)$

$$\left.\begin{array}{l}\text{NEB: Normalized Empirical Bias}\\ \text{NESD: Norm. Emp. Standard Deviation}\end{array}\right\} \Rightarrow \begin{array}{l}\text{estimated over}\\ \text{60 000 simulations}\end{array}$$

θ^*: MLE estimator when the B_i are known

Initial intensity: $\lambda(t) = \alpha\beta t^{\beta-1}$ or $\lambda(t) = \beta/\eta(t/\eta)^{\beta-1}$

EM is more robust than direct likelihood maximization

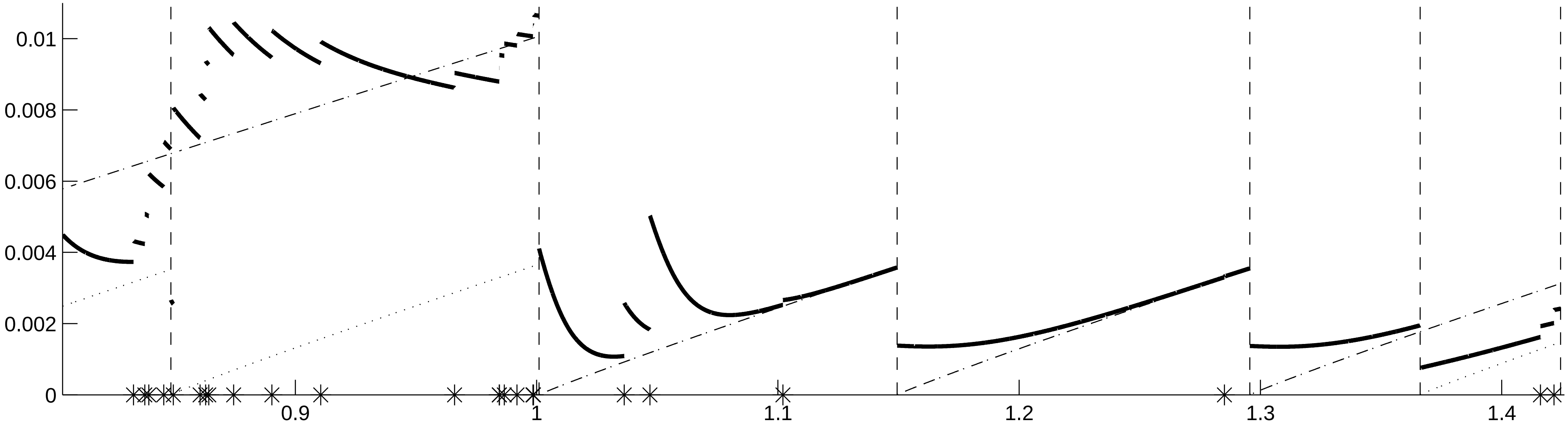
	NEB				NESD			
	α	η	β	p	α	η	β	p
$\tilde{\theta}$	590%	**36%**	**8.2%**	**-9.4%**	3300%	130%	**35%**	**50%**
$\hat{\theta}$	**580%**	37%	9.2%	-12%	3300%	130%	36%	51%
θ^*	18000%	-24%	-12%	0%	5000%	35%	22%	25%

Quality of individual PM efficiency estimation:

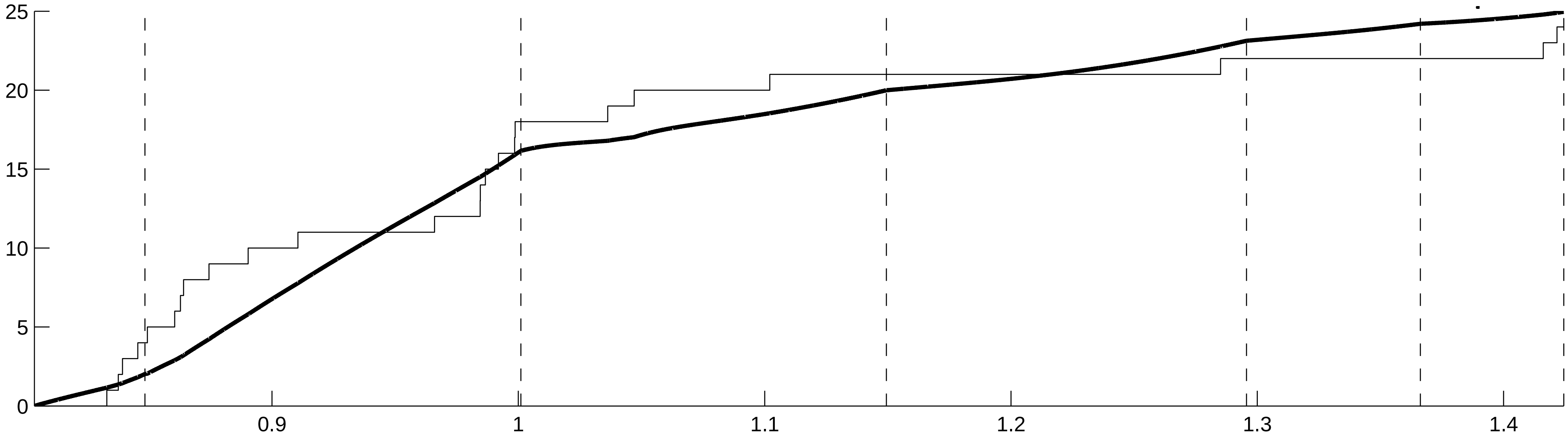
	Empirical mean		Empirical standard deviation	
m^{th} PM	$\pi_m^{\tilde{\theta}} - B_m$	$\tilde{p} - B_m$	$\pi_m^{\tilde{\theta}} - B_m$	$\tilde{p} - B_m$
1	-7.8 e-2	**-5.4 e-2**	5.8 e-1	5.8 e-1
2	-9.3 e-2	**-5.8 e-2**	5.9 e-1	**5.8 e-1**
3	-9.4 e-2	**-5.8 e-2**	5.9 e-1	**5.7 e-1**
4	-6.8 e-2	**-5.8 e-2**	6.0 e-1	**5.6 e-1**
5	**-4.3 e-2**	-5.6 e-2	5.3 e-1	**5.2 e-1**
6	-3.7 e-1	**-6.1 e-2**	6.4 e-1	**5.8 e-1**
7	**-1.3 e-2**	-5.8 e-2	**2.8 e-1**	4.6 e-1
8	**7.2 e-3**	-5.8 e-2	**3.9 e-1**	5.1 e-1
9	**9.4 e-4**	-5.4 e-2	**4.0 e-1**	5.2 e-1
10	**-8.5 e-3**	-5.5 e-2	**4.7 e-1**	5.5 e-1
Mean	-7.5 e-2	**-5.7 e-2**	**5.1 e-1**	5.4 e-1

$\Rightarrow$ **Individual PM efficiency estimation is relevent after** c

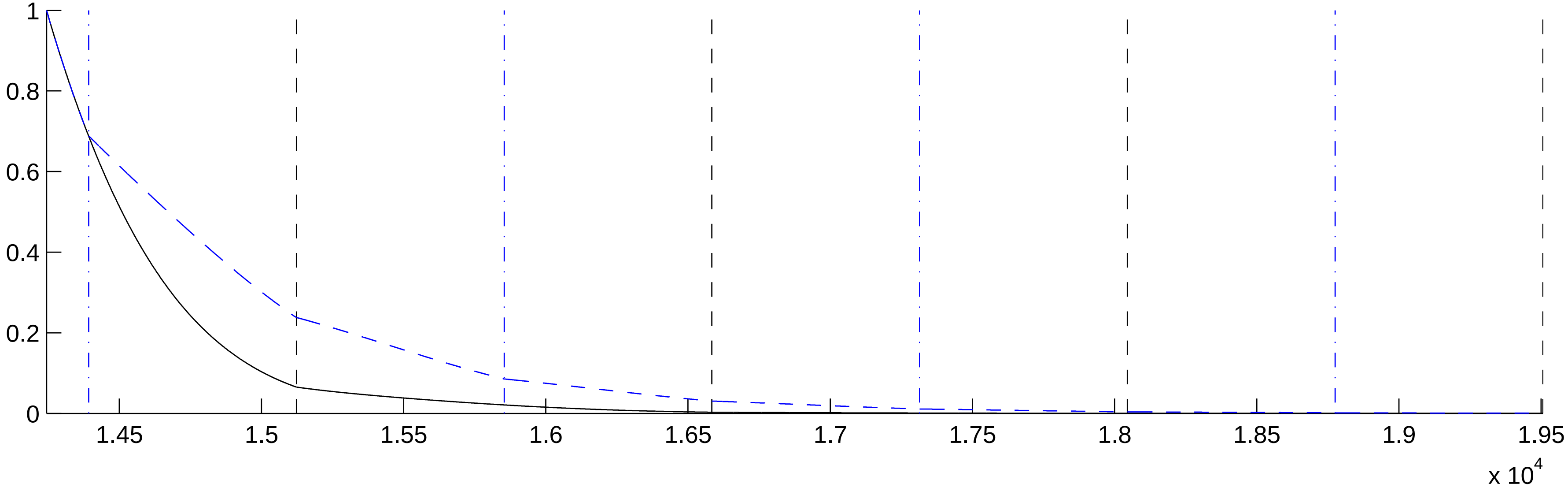
Failure intensity for $\theta = \hat{\theta}$



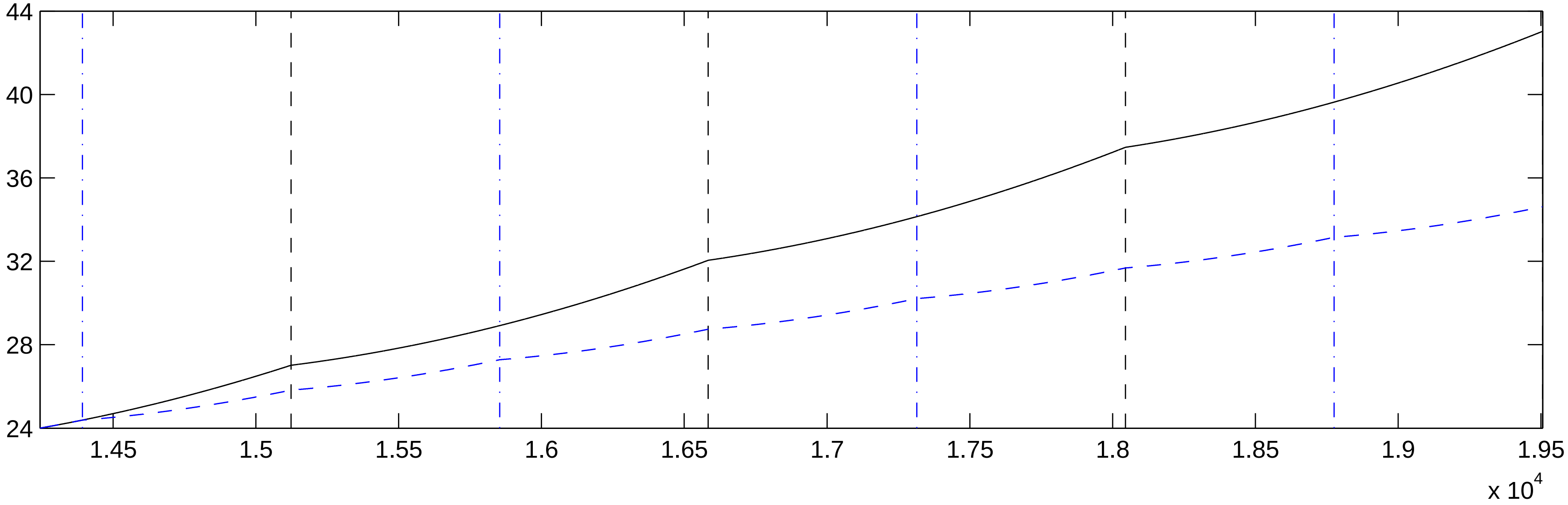
Cumulative failure intensity and number of failures for $\theta = \hat{\theta}$



Forecast reliability for $\theta = \hat{\theta}$



Forecast cumulative number of failures for $\theta = \hat{\theta}$



VIII. Prospects

- Maintenance times optimization
- Consider a more general distribution over $]-\infty, 1]$ for the B_i.
- Develop confidence intervals and tests for the BP model
- ...

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