Degradation analysis based on the Inverse Gaussian Process model

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- Research background
- Degradation modelling
- Numerical experiments
- Conclusion

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PART ONE

Research background

Reliability is defined as (ISO):

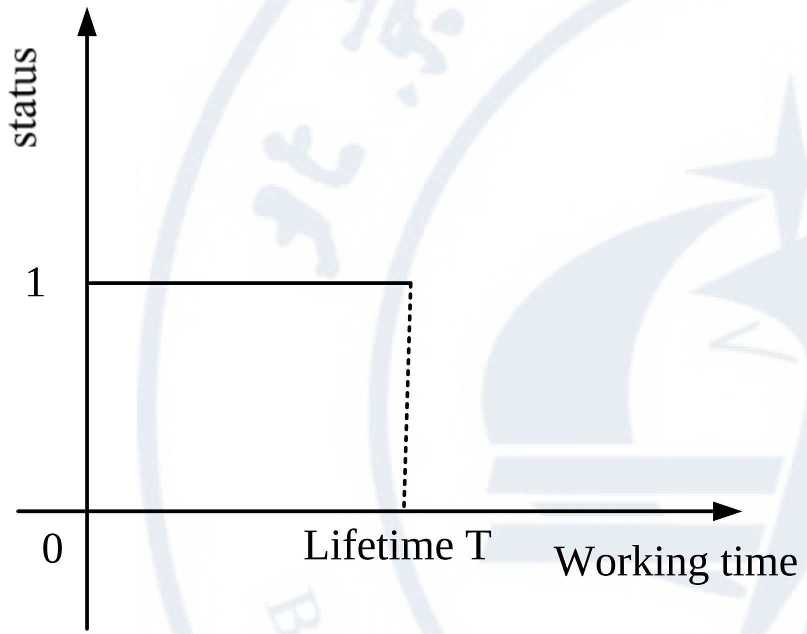
- The ability of an item to perform a required function, under given environmental and operating conditions and for a stated period of time.

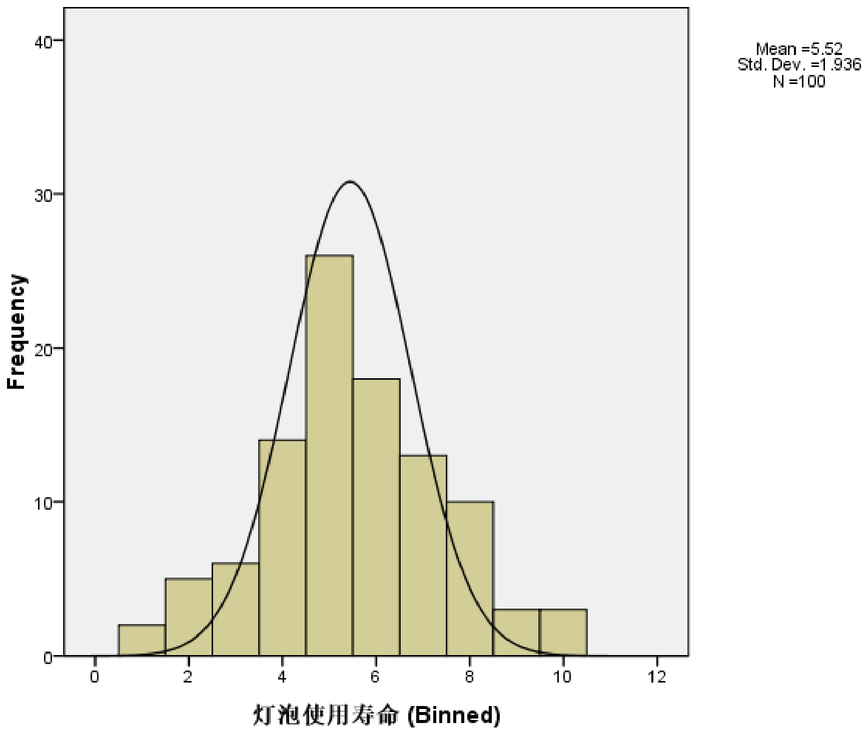
Failure: the item loses this ability.

Reliability data analysis:

- Lifetime data;
- degradation measurement.

Lifetime data analysis





Lifetime data collection → Lifetime distribution selection → System reliability assessment

Degradation analysis

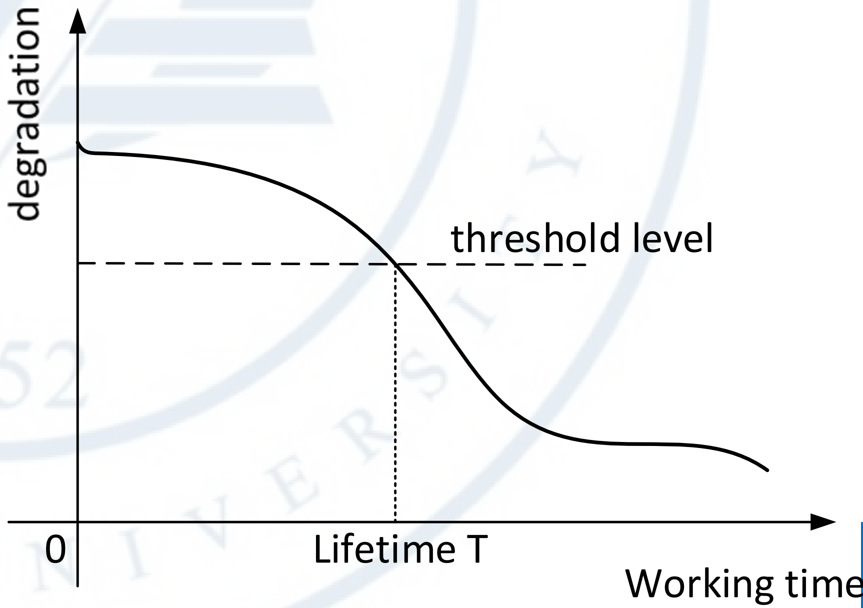
New challenges:

- High-reliability and long-life products
- Temporal variability and unit-to-unit variability
- Dynamic decision-making in maintenance

Reliability analysis based on degradation data:

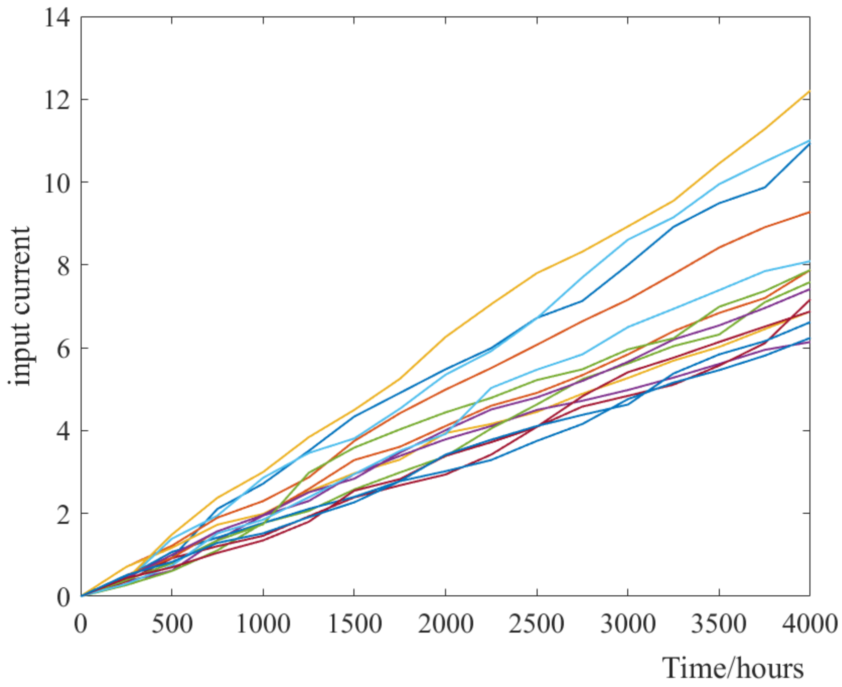
$$R(t) = P\{X(\tau) \notin \Omega, \forall 0 \le \tau \le t\}$$

$$T = \inf\{t : X(t) \in \Omega\}$$



- Statistically independent degradation increments;
- $Y_0(t) - Y_0(s)$ follows an IG distribution $IG\left(\beta\Delta\Lambda(t), \eta\Delta\Lambda^2(t)\right)$.

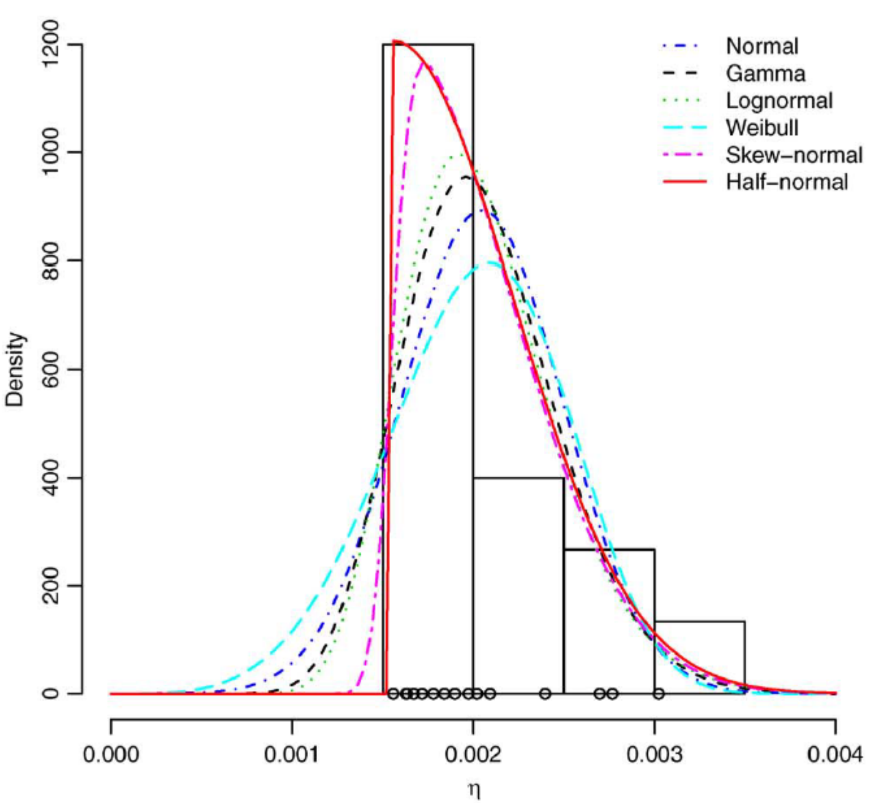
✓ better fitting results for the GaAs laser degradation.



Wang & Xu (2010): An Inverse Gaussian Process Model for Degradation Data

Symmetric and normal random effects

$$X_0(t) - X_0(s) \sim IG\left(\beta\Delta\Lambda(t), \eta\Delta\Lambda^2(t)\right), \beta^{-1} \sim N(\mu, \sigma^2)$$



Distribution type	L	AIC
Normal	-89.8168	183.63
Gamma	-90.3049	184.61
Lognormal	-90.6636	185.33
Weibull	-89.1474	182.29
Skew-normal	**-87.4892**	**178.98**

Problems for the IG process

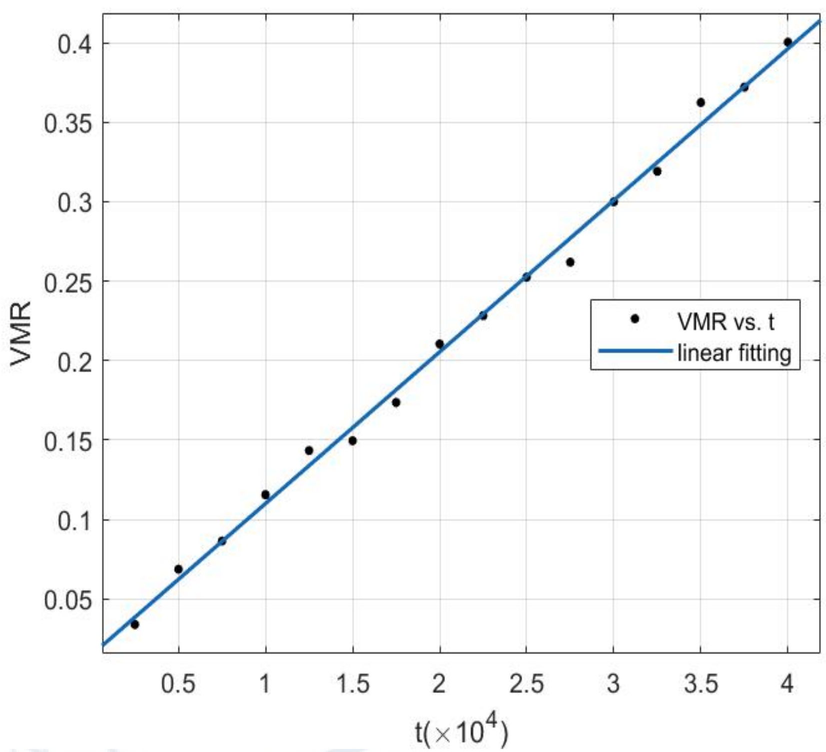
□ Constant variance-to-mean ratio (VMR)

For the additivity of the IG distributions $IG\left(\beta\Delta\Lambda(t), \eta\Delta\Lambda^2(t)\right)$.

$$E[X(t)] = \beta\Lambda(t)$$

$$Var[X(t)] = \frac{[\beta\Lambda(t)]^3}{\eta\Lambda^2(t)} = \frac{\beta^3}{\eta}\Lambda(t)$$

$$VMR[X(t)] = \frac{Var[X(t)]}{E[X(t)]} = \frac{\beta^2}{\eta}$$



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PART TWO

Degradation modelling

① SN random effects ② time varying VMR

Degradation model $\{X_0(t), t \geq 0\}$:

- Statistically independent degradation increments;
- Degradation level $X_0(t)$ follows IG distribution:

$$F_{X_0}(x) = \Phi\left[\sqrt{\frac{\eta\Lambda^2(t)}{x}}\left(\frac{x}{\beta\Lambda(t)} - 1\right)\right] + \exp\left\{\frac{2\eta\Lambda(t)}{\beta}\right\}\Phi\left[-\sqrt{\frac{\eta\Lambda^2(t)}{x}}\left(\frac{x}{\beta\Lambda(t)} + 1\right)\right]$$

- $\delta = \beta^{-1}$ follows skew-normal distribution $SN(\mu, \sigma^2, \alpha)$:

$$f_\delta(u) = \frac{2}{\sigma}\phi\left(\frac{u-\mu}{\sigma}\right)\Phi\left(\alpha\frac{u-\mu}{\sigma}\right)$$

Reliability assessment:

$$R(t) = P\{X(\tau) \notin \Omega, \forall 0 \leq \tau \leq t\}, T = \inf\{t: X(t) \in \Omega\}$$

Considering IG process is monotonous, its reliability can be simplified by:

$$F_{T_0}(t) = P\{X_0(t) > D\} = 1 - F_{X_0}(D)$$

If we consider SN random effects, lifetime CDF will be:

$$F_{T_2}(t) = \int_{-\infty}^{+\infty} F_{T_0}(t|u) f_\delta(u)\, du$$

$$= E_\delta\left[\Phi\left(-\sqrt{\frac{\eta\Lambda^2(t)}{x}} + \sqrt{\eta x}\delta\right)\right] + E_\delta\left[e^{2\eta\Lambda(t)\delta}\Phi\left(-\sqrt{\frac{\eta\Lambda^2(t)}{x}} - \sqrt{\eta x}\delta\right)\right]$$

Reliability assessment:

For $V \sim SN(\mu, \sigma^2, \alpha)$, the computations of $E_V[\Phi(A+BV)]$ and $E_V\,[e^{CV}\Phi(A+BV)]$:

(1) $I(a_1, a_2, b_1, b_2) = \int_{-\infty}^{+\infty} \Phi(a_1 + b_1 x)\Phi(a_2 + b_2 x)\phi(x)\, dx$;

(2) Based on Dominated Convergence Theorem, compute the second order partial derivative of $I(a_1, a_2, b_1, b_2)$ with respect to a_1, a_2;

(3) Integrate it over a_1, a_2;

(4) With some changes of variables, obtain the results.

Parameter estimation:

Test data: $x_{n,m} = x(t_{n,m})$ denotes the m^{th} degradation measurement of sample n; $\Delta x_{n,m} = x_{n,m} - x_{n,m-1}$.

Log-Likelihood function: $L(x) = \sum_{n=1}^{N} \sum_{m=1}^{M} \ln f_{\Delta X}(\Delta x_{n,m})$

where

$$f_{\Delta X}(x) = \int_{-\infty}^{+\infty} f_{\Delta X_0}(x|\delta = \beta^{-1} = u) f_\delta(u)\, du$$

$$= E_\delta\left[\sqrt{\frac{\eta \Delta\Lambda^2(t)}{2\pi x^3}} \exp\left\{-\frac{\eta}{2x}\left[x^2\delta^2 - 2\Delta\Lambda(t) x\delta + \Delta\Lambda^2(t)\right]\right\}\right]$$

Maximize $L(x)$ and obtain the MLEs of $\Theta = \{\theta_\Lambda, \eta, \mu, \sigma, \alpha\}$.

a perturbed IG process model $\{Y(t), t \geq 0\}$:

$$Y_k = X_k + \varepsilon_k$$

where $Y_k = Y(t_k)$, $X_k = X(t_k)$, $\varepsilon_k = \varepsilon(t_k)$ are respectively the measured degradation, actual degradation and measurement error (ME) at t_k.

- Statistically independent ME: $\varepsilon_k \sim N(0, \sigma_\varepsilon^2)$:

$$E[Y(t)] = \beta\Lambda(t)$$

$$Var[Y(t)] = \frac{\beta^3}{\eta}\Lambda(t) + \sigma_\varepsilon^2$$

$$VMR[Y(t)] = \frac{\beta^2}{\eta} + \frac{\sigma_\varepsilon^2}{\beta\Lambda(t)}$$

a perturbed IG process model $\{Y(t), t \geq 0\}$:

$$Y_k = X_k + \varepsilon_k$$

- Statistically dependent ME: $\varepsilon_k \sim N(0, \sigma_{\varepsilon_k}^2(x_k))$:

$$f_{\varepsilon_k|X_k}(z|x_k) = \frac{1}{\sqrt{2\pi}\sigma_{\varepsilon_k}(x_k)} exp\left[-\frac{z^2}{2\sigma_{\varepsilon_k}^2(x_k)}\right]$$

- Reliability assessment:

$$F_T(t) = P\{X(t) < D\}$$
$$= \Phi\left[\sqrt{\frac{\eta\Lambda^2(t)}{D}}\left(\frac{D}{\beta\Lambda(t)} - 1\right)\right] + \exp\left\{\frac{2\eta\Lambda(t)}{\beta}\right\}\Phi\left[-\sqrt{\frac{\eta\Lambda^2(t)}{D}}\left(\frac{D}{\beta\Lambda(t)} + 1\right)\right]$$

VMR for the perturbed IG process model:

$$VMR(Y_k) = \frac{\beta^2}{\eta} + \frac{\sqrt{\eta}}{\sqrt{2\pi}\beta}\int_0^{+\infty} \sigma_{\varepsilon_k}^2(x_k) x^{-\frac{3}{2}} e^{-\frac{\eta[x-\beta\Lambda(t_k)]^2}{2\beta^2 x}} dx$$

(1) On condition of X_k, Y_k follows a normal distribution:

$$f_{Y_k|X_k}(y|x_k) = f_{\varepsilon_k|X_k}(y - x_k|x_k)$$

(2) Conditional expectation and variance:

$$E[Y_k|X_k = x_k] = x_k,\ Var(Y_k|X_k = x_k) = \sigma_{\varepsilon_k}^2(x_k)$$

(3) Unconditional expectation and variance:

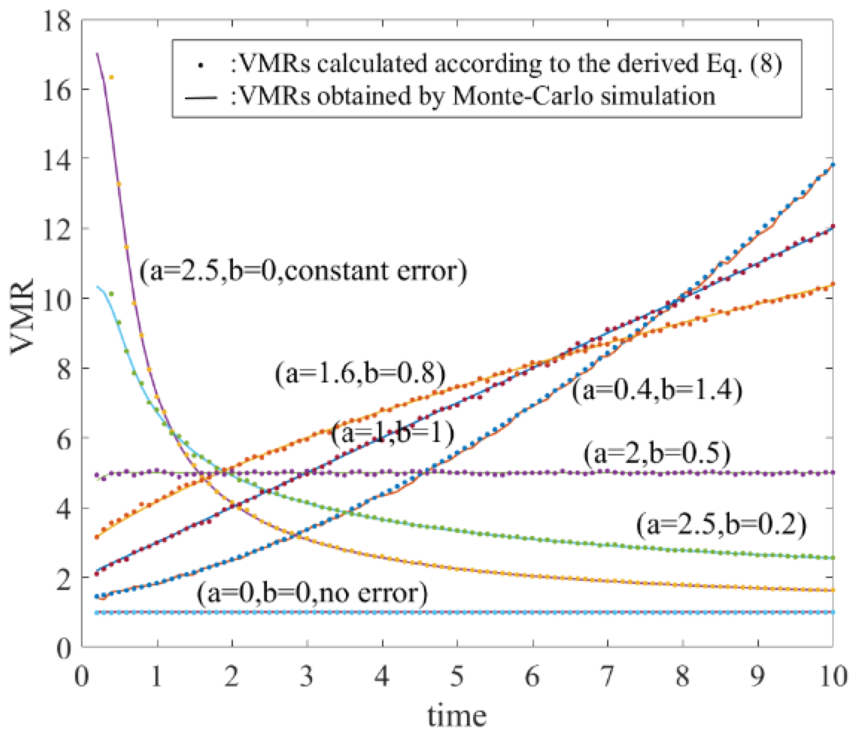
$$E(Y_k) = E_X[E_Y(Y_k|x_k)] = \beta\Lambda(t_k)$$

$$Var(Y_k) = E_X[Var(Y_k|x_k)] + Var_X[E(Y_k|x_k)]$$

VMR for the perturbed IG process model:

Assume a flexible power function: $\sigma_{\varepsilon_k}(x_k) = a{x_k}^b, a, b > 0.$

Depending on different values of a, b, the VMR of the proposed model can display various kinds of trends:



Parameter estimation:

The introduction of ME makes that the measured degradation increments are no longer independent.

Based on the concept of conditional probability, the likelihood function can be expressed as:

$$L(y;\Theta)=\prod_{i=1}^{n} f_{Y_{i,1}}(y_{i,1})\, f_{Y_{i,2}|Y_{i,1}}(y_{i,2}|y_{i,1})\cdots f_{Y_{i,m_i}|Y_{i,m-1}}(y_{i,m_i}|y_{i,m_i-1})$$

where $y_{i,k}=y(t_{i,k})$ is the measured degradation performance, and $f_{Y_{i,k}|Y_{i,k-1}}(y_{i,k}|y_{i,k-1})$ is the PDF of $y_{i,k}$ conditioned on $y_{i,k-1}$.

Parameter estimation:

The conditional PDF $f_{Y_{i,k}|Y_{i,k-1}}(y_{i,k}|y_{i,k-1})$ can be computed in the following step-by-step way.

$Y_{i,k-1}$ → $X_{i,k-1}$ → $X_{i,k}$ → $Y_{i,k}$

(1) $f_{X_{i,k-1}|Y_{i,k-1}}(v|y_{i,k-1}) = \dfrac{f_{Y_{i,k-1}|X_{i,k-1}}(y_{i,k-1}|v)\cdot f_{X_{i,k-1}}(v)}{\int_0^{+\infty} f_{Y_{i,k-1}|X_{i,k-1}}(y_{i,k-1}|u) f_{X_{i,k-1}}(u)du}$

(2) $f_{X_{i,k}|Y_{i,k-1}}(w|y_{i,k-1}) = \int_0^{+\infty} f_{X_{i,k}|X_{i,k-1}}(w|v)\, f_{X_{i,k-1}|Y_{i,k-1}}(v|y_{i,k-1})dv$

(3) $f_{Y_{i,k}|Y_{i,k-1}}(y_{i,k}|y_{i,k-1}) = \int_0^{+\infty} f_{Y_{i,k}|X_{i,k}}(y_{i,k}|w)\, f_{X_{i,k}|Y_{i,k-1}}(w|y_{i,k-1})dw$

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PART THREE

Numerical experiments

① simulation study ② two case studies

Performance of the parameter estimation methods:

- Power drift function: $\Lambda(t) = t^b$
- 1000 repetitions to calculate bias and standard deviations

➢ The MLEs can accurately estimate the model parameters

➢ The estimation accuracy increases with sample size

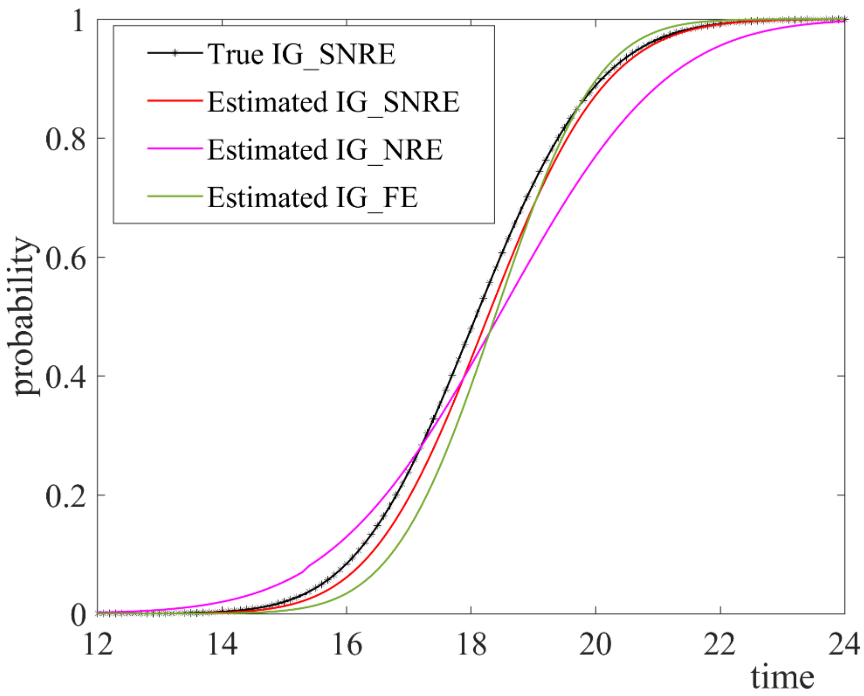
Sample size	$\mu = 2$	$\sigma = 0.2$	$\alpha = 5$	$\eta = 2$	$b = 1.2$
30	0.0747 (0.1441)	-0.0201 (0.0343)	0.0261 (0.2459)	0.0143 (0.2655)	0.0035 (0.0174)
60	0.0713 (0.1435)	-0.0149 (0.0271)	0.0205 (0.2430)	0.0119 (0.2620)	0.0032 (0.0161)
90	0.0460 (0.1397)	-0.0128 (0.0264)	0.0235 (0.2553)	0.0078 (0.2517)	0.0015 (0.0162)

Lifetime estimation accuracy with mis-specification:

① Simulate degradation data under the proposed model

② Estimate and compare three candidate models

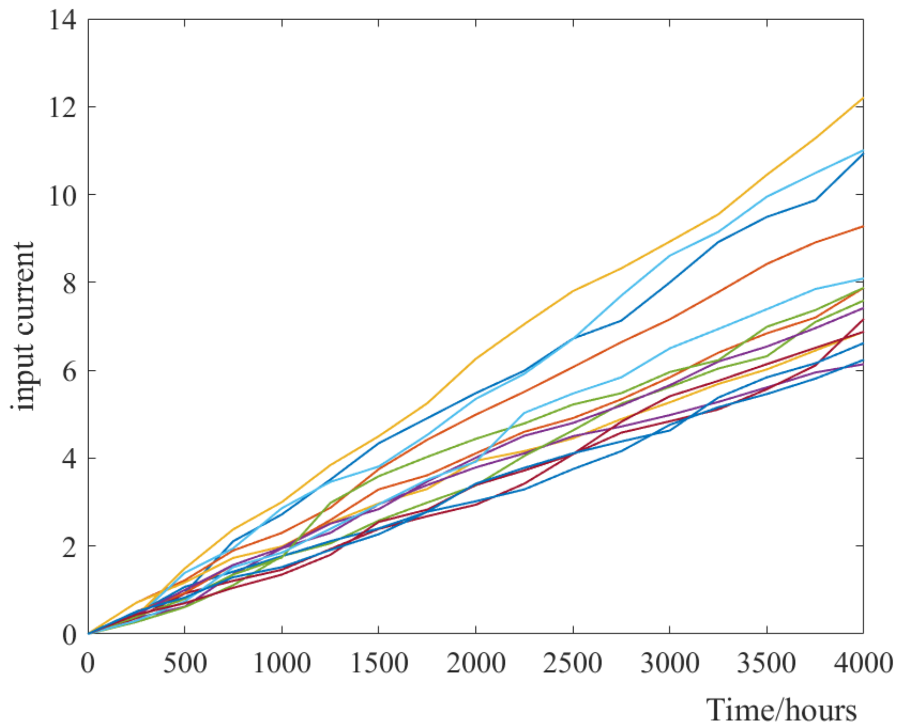
- the true model can be selected with the lowest MSE



Candidate Model	MSE
IG_FE	1.91E-3
IG_NRE	3.62E-3
IG_SNRE	**5.68E-04**

GaAs laser device degradation experiment:

- Sample size: 15
- Inspection interval: 250 hours.
- Failure threshold level: 10A



Parameter estimation results – random effects:

A linear drift function is chosen based on experience.

Compared to models IG_FE and IG_NRE, the proposed EIG model with SN random effects has the smallest AIC, and therefore has the best degradation fitting performance.

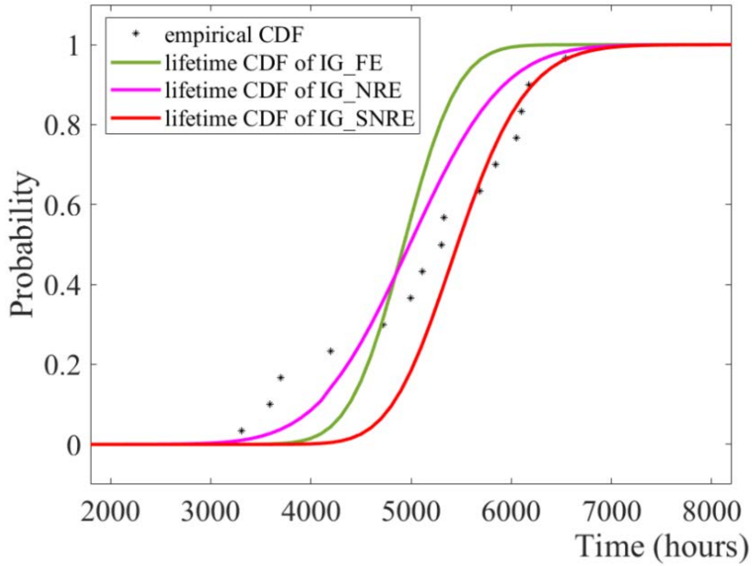
	η	μ	σ	α	L_{max}	*AIC*
IG_FE	5.43E-5	490.65	-	-	75.03	-146.06
IG_NRE	6.09E-5	498.59	61.30	-	75.23	-144.46
IG_SNRE	6.09E-5	500.26	61.41	-29.70	**77.18**	**-146.36**

Reliability assessment results – random effects :

- Present the empirical CDF of the 15 samples as dots.
- Plot the lifetime CDFs for the candidate models as lines.
- Calculate the MSEs of candidate models with the empirical CDF.

The proposed model IG_SNRE can most precisely estimate the lifetime distribution.

Candidate Model	MSE
IG_FE	5.21E-3
IG_NRE	2.22E-3
IG_SNRE	**1.62E-3**



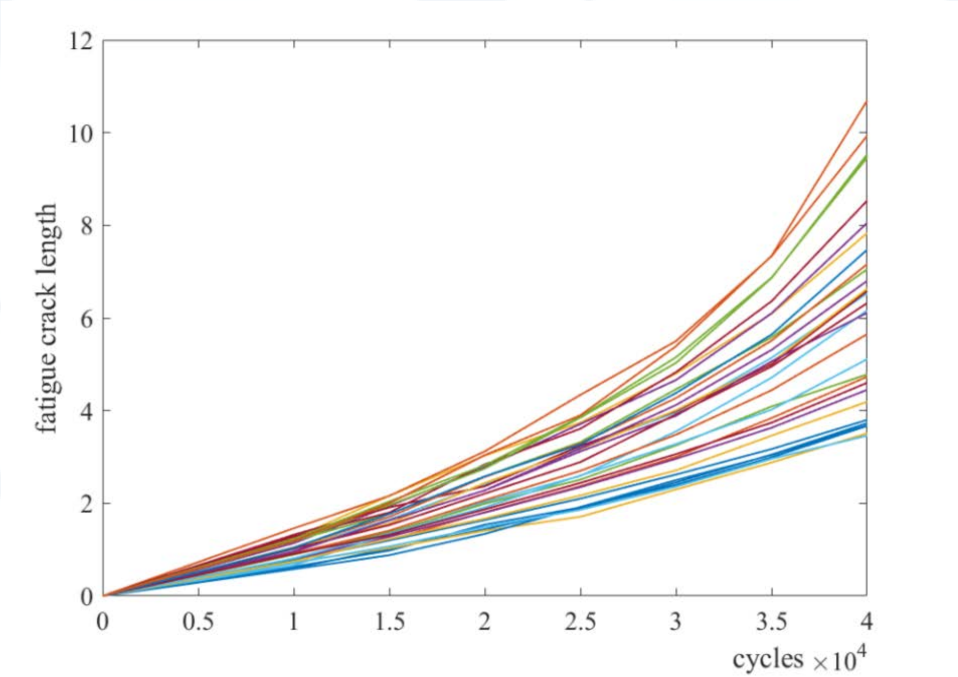
Parameter estimation results – measurement errors:

Compared to the existing models IG_0 and IG_SIME, the proposed model IG_SDME with statistically dependent measurement error has the smallest AIC, and therefore has the best degradation fitting performance.

	η	β	a	b	L_{max}	AIC
IG_0	5.43E-5	2.04E-3	-	-	75.03	-146.06
IG_SIME	5.44E-5	2.04E-3	9.00E-4	-	75.02	-144.04
IG_SDME	5.12E-5	2.04E-3	2.96E-3	0.9917	**77.69**	**-147.38**

Constant amplitude fatigue test:

- Sample size: 30
- Measurement interval: first after 10000 cycles and then every 5000 cycles.
- Failure threshold level: 15



Analysis of random effects:

An exponential drift function is chosen.

5 candidate distribution forms are compared for the inverse slopes of 30 specimens, among which the skew-normal distribution performs the best.

Distribution type	L	AIC
Normal	24.6104	-45.2208
Gamma	25.4566	-46.9132
Lognormal	24.3753	-44.7506
Weibull	25.5983	-47.1966
Skew-normal	**26.8775**	**-47.755**

Parameter estimation results:

The proposed model IG_SNREME, considering both skew-normal random effects and measurement errors, has the smallest AIC and best degradation fitting performance.

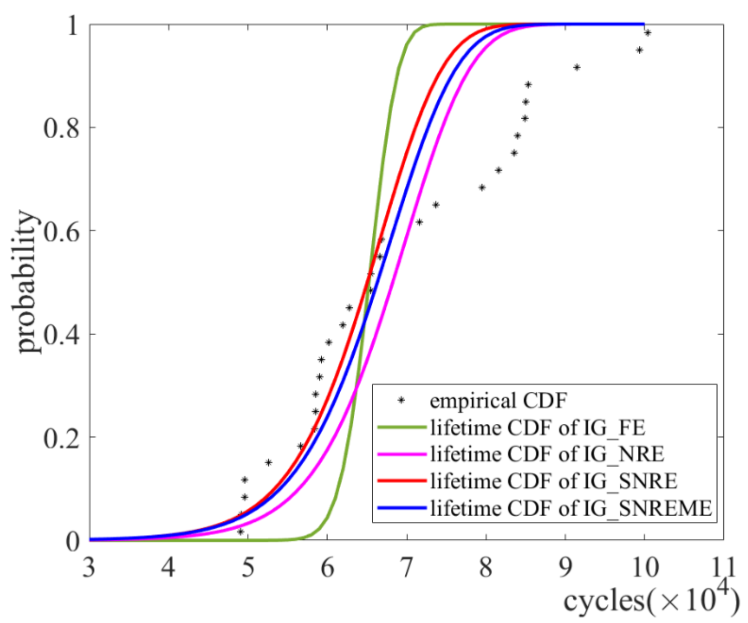
	η	μ	σ	α	γ	σ_ε	L_{max}	AIC
IG_FE	92.69	0.2915	-	-	0.2582		-41.49	88.99
IG_NRE	204.17	0.2829	0.066	-	0.2428		-37.01	82.03
IG_SNRE	136.73	0.3093	0.077	0.8252	0.2680		-33.70	77.40
IG_SNREME	186.55	0.2805	0.070	0.8481	0.2509	2.54E-4	**-32.63**	**77.26**

Reliability assessment results:

- Present the empirical CDF of the 30 specimens as dots.
- Plot the lifetime CDFs for the candidate models as dashes.
- Calculate the MSEs of candidate models with the empirical CDF.

The proposed model IG_SNREME can most precisely estimate the lifetime distribution.

Candidate Model	MSE
IG_FE	4.53E-3
IG_NRE	2.14 E-3
IG_SNRE	1.70 E-3
IG_SNREME	**1.66 E-3**



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PART FOUR

Conclusion

What has been done

- ✓ Extend the traditional IG process by incorporating skew-normal random effects;
- ✓ Analytically assess the reliability for the EIG process;
- ✓ Propose a perturbed IG process model with statistically dependent measurement error;
- ✓ Obtain the VMR of the perturbed IG process model;
- ✓ Verify the two degradation models by numerical experiments of GaAs laser and fatigue crack growth.

What can be done in the future

- Combination of both skew-normal random effects and statistically dependent measurement errors;
- Degradation test planning and maintenance strategy optimization for the extended and perturbed IG process;
- Generalization of the IG process to have time-varying VMR.

Thanks for listening!

Songhua Hao